

On numerical solutions to stochastic Volterra equations¹

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Abstract

The aim of the paper is to demonstrate the use of the Galerkin method for some kind of Volterra equations, deterministic and stochastic as well. The paper consists of two parts: the theoretical and numerical one. In the first part we recall some apparently well-known results concerning the Volterra equations under consideration. In the second one we describe a numerical algorithm used and next present some examples of numerical solutions in order to illustrate the pertinent features of the technique used in the paper.

Key words: Stochastic and deterministic Volterra equations, Galerkin method.

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1 Introduction

In the paper we investigate a stochastic version of a linear Volterra equation of the general form

$$X(t, x) = \int_0^t a(t - \tau) A X(\tau, x) d\tau + X_0(x) + f(t, x), \quad (1)$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $a \in L^1_{loc}(\mathbb{R}_+)$, A is a linear operator and f some mapping. To fix our attention we shall consider the equation (1) in a separable Hilbert space H with a scalar product (\cdot, \cdot) , a norm $|\cdot|$ and a complete orthonormal system $\{e_n\}$. The equation (1) creates a big class of equations and generalizes heat and wave equations and even linear Navier-Stokes equation. We refer to the excellent monograph [13] for a rich survey. That kind of Volterra equation has been studied by many authors in connection with problems arising in mathematical physics, particularly in viscoelasticity, heat conduction in materials with memory, energy balance and termoviscoelasticity. In order to take into account random fluctuations, we have to consider the equation (1) with random external force.

There are our first considerations concerning numerical treatment of stochastic Volterra equations, so we will be grateful for readers' remarks and advices.

Next, we plan to study the probabilistic features of family of trajectories, take into account different noises and develop numerical schemes for cases where the analytic form of resolvent is not known.

2 Resolvent approach

Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a probability space with a complete right-continuous filtration and $W(t)$, $t \geq 0$, is a cylindrical Wiener process with values in the space H . Let us omit, for convenience, the space variable x in the equation (1) and introduce the process $W(t)$, $t \geq 0$, instead of function f . Hence, we arrive at the following stochastic Volterra equation

$$X(t) = \int_0^t a(t - \tau) A X(\tau) d\tau + X_0 + W(t). \quad (2)$$

In this part of the paper we recall some results concerning solutions to (2). We restrict our considerations to paper containing so-called resolvent approach to

Volterra equation. The notion *resolvent* or *fundamental solution* for Volterra equation (1) probably comes from Friedman and Shinbrot [5] who studied deterministic Volterra integral equations in Banach space. For recent survey we refer again to [13]. In the sequel we shall assume that the equation (1) is *well-posed*, that is, that (1) admits resolvent $S(t)$, $t \geq 0$.

As in deterministic case, the *mild solution* to the stochastic Volterra equation (2) is of the form

$$X(t) = S(t)X_0 + \int_0^t S(t-\tau)dW(\tau), \quad t \geq 0, \quad (3)$$

where $S(t)$, $t \geq 0$, is the resolvent family for the equation (1) determined by the operator A and the function a . In order to study solution (3) it is enough to consider the stochastic convolution

$$W_S(t) := \int_0^t S(t-\tau)dW(\tau), \quad t \geq 0, \quad (4)$$

where the stochastic integral is defined according to particular case under consideration.

Stochastic Volterra equations with resolvent approach have been treated by several authors, see e.g. [1], [2], [3], [4], [11] and recently [8] and [9]. In the first three papers stochastic Volterra equations are studied in connection with viscoelasticity and heat conduction in materials with memory. The paper [1] is particularly significant because the authors were the first who have extended the well-known semigroup approach, applied to stochastic differential equations, to the equation (2). The resolvent approach is a natural way of extension the semigroup approach which is well-known from the theory of evolution equations. That approach enables to follow some results and schemes obtained for semigroups. Unfortunately, some results are not valid in our case because resolvent family, $S(t)$, $t \geq 0$, does not satisfy semigroup property.

Clément and DaPrato studied stochastic Volterra equation (2) where A was self-adjoint, negative operator in the space H , such that

$$Ae_k = -\mu_k e_k, \quad \mu_k > 0, \quad k \in \mathbb{N}.$$

They considered stochastic Volterra equation (2) driven by the noise term W of the form

$$(W(t), h)_H = \sum_{k=1}^{+\infty} (h, e_k)_H \beta_k(t), \quad h \in H, \quad (5)$$

where $\{\beta_k\}$ was a sequence of real-valued, independent Wiener processes. They assumed that the kernel function a is completely positive. The consequence of completely positiveness of the function a is that the solution $s(\cdot, \gamma)$, $\gamma > 0$, to the following equation

$$s(t) + \gamma \int_0^t v(t - \tau) s(\tau) d\tau = 1, \quad t \geq 0, \quad (6)$$

is nonnegative and nonincreasing for any $\gamma > 0$. In fact, $s(t) \in [0, 1]$. In [1] regularity of stochastic convolution (4) is studied and h lderianity of the corresponding trajectories is proved.

Hypothesis 1

- (i) A is a self-adjoint negative operator and $Ae_k = -\mu_k e_k$.
- (ii) a is completely positive.
- (iii) $-\text{Tr}(A^{-1}) = \sum_{k=1}^{\infty} (1/\mu_k) < +\infty$.

Hypothesis 2 *There exists $\theta \in (0, 1)$ and $C_\theta > 0$ such that, for all $0 < \tau < t$ we have*

$$\int_{\tau}^t s^2(\mu, \sigma) d\sigma \leq C_\theta \mu^{\theta-1} |t - \tau|^\theta,$$

$$\int_0^{\tau} [s(\mu, \tau - \sigma) - s(\mu, t - \sigma)]^2 d\sigma \leq C_\theta \mu^\theta |t - \tau|^\theta$$

and

$$\sum_{k=1}^{\infty} \mu_k^{\theta-1} < +\infty.$$

Hypothesis 3 *There exists $M > 0$ such that*

$$\begin{cases} |e_k(\theta)| \leq M, & k \in \mathbb{N}, \quad \theta \in \mathcal{O}, \\ |\nabla e_k(\theta)| \leq M \mu_k^{1/2}, & k \in \mathbb{N}, \quad \theta \in \mathcal{O}, \end{cases}$$

where \mathcal{O} is a bounded open subset of \mathbb{R}^d .

Cl ment and DaPrato proved the following results.

Theorem 1 (*[1], Theorem 2.2*) *Assume that Hypothesis 1 holds. Then for any $t \geq 0$ the series*

$$\sum_{k=1}^{\infty} \int_0^t s(\mu_k, t - \tau) e_k d\beta_k(\tau),$$

is convergent in $L^2(\Omega)$ to a Gaussian random variable $W_S(t)$ with mean 0 and covariance operator Q_t determined by

$$Q_t e_k = \int_0^t s^2(\mu_k, \tau) d\tau e_k, \quad k \in \mathbb{N}.$$

Theorem 2 ([1], Proposition 3.3) *Under Hypotheses 1 and 2, for every positive number $\alpha < \theta/2$, the trajectories of W_S are almost surely α -Hölder continuous.*

Theorem 3 ([1], Theorem 4.1) *Under Hypotheses 1, 2 and 3, the trajectories of W_S are almost surely α -Hölder continuous in (t, x) for any $\alpha \in (0, 1/4)$.*

In the paper [2], white noise perturbation of an integro-differential equation arising in the study of evolution of material with memory is studied. In the next paper [4], the authors considered evolutionary integral equations as appearing in the theory of linear parabolic viscoelasticity forced by white noise. As earlier, they studied the stochastic convolution that provides regular solutions. Additionally, under suitable assumptions the authors proved that the samples are Hölder-continuous. In the remaining part of the paper [4], the results obtained of that paper were put in a wider perspective by consideration of equations with fractional derivatives.

In the paper [3], the authors first proved that $W_S(t)$ is a Gaussian random variable for any $t \geq 0$. Next, the transition function $P_t, t \geq 0$, associated with $S(t)$ was considered. When $S(t)$ is the resolvent operator of a stochastic Volterra equation, the convolution $W_S(t), t \geq 0$, is not a Markov process. This fact has the consequence that $P_t, t \geq 0$, is not a semigroup and then it is not possible to associate to P_t a Kolmogorov equation. However, the authors characterized those transition functions such that $P_t \varphi$ was differentiable for any uniformly continuous and bounded function φ .

There are some other regularity results concerning stochastic convolution (4). In [11] was studied the case when the equation (2) was driven by a correlated, spatially homogeneous Wiener process W with values in the space of real, tempered distributions $S'(\mathbb{R}^d)$. Let Γ be the covariance of $W(1)$ and the associated spectral measure be μ . We considered existence of the solutions to (2) in $S'(\mathbb{R}^d)$ and derived conditions under which the solutions to (2) were function-valued and continuous. In that case, the initial value $X_0 \in S'(\mathbb{R}^d)$, a is a locally integrable function and A is an operator given in the Fourier transform form

$$\mathcal{F}(A\xi)(\lambda) = -\beta(\lambda)\mathcal{F}(\xi)(\lambda). \quad (7)$$

We introduce the following hypothesis.

Hypothesis 4 (1) For any $\gamma \geq 0$, the equation (6) has exactly one solution $s(\cdot, \gamma)$ locally integrable and measurable with respect to both variables $\gamma \geq 0$ and $t \geq 0$.

(2) Moreover, for any $T \geq 0$, $\sup_{t \in [0, T]} \sup_{\gamma \geq 0} |s(t, \gamma)| < +\infty$.

For some special cases the function $s(t; \gamma)$ may be found explicitly. For instance

$$\begin{aligned} \text{for } a(t) &\equiv 1, & s(t; \gamma) &= e^{-\gamma t}, & t \geq 0, \gamma \geq 0; \\ \text{for } a(t) &= t, & s(t; \gamma) &= \cos(\sqrt{\gamma}t), & t \geq 0, \gamma \geq 0; \\ \text{for } a(t) &= e^{-t}, & s(t; \gamma) &= (1 + \gamma)^{-1}[1 + \gamma e^{-(1+\gamma)t}], & t \geq 0, \gamma \geq 0. \end{aligned} \quad (8)$$

In that case, the resolvent family $S(t)$, $t \geq 0$, determined by the operator A and the function a is given by the formula (7) and has the form

$$S(t)\xi = \mathbf{r}(t) \star \xi, \quad \xi \in S'(\mathbb{R}^d),$$

where $\mathbf{r}(t) = \mathcal{F}^{-1}s(t, \beta(\cdot))$, $t \geq 0$. The following results for stochastic convolution are consequences of properties of stochastic integral.

Theorem 4 ([11], Theorem 2) Let W be a spatially homogeneous Wiener process and $S(t)$, $t \geq 0$, the resolvent for the equation (2). If Hypothesis 4 holds then the stochastic equation

$$S \star W(t) = \int_0^t S(t - \sigma) dW(\sigma), \quad t \geq 0$$

is a well-defined $S'(\mathbb{R}^d)$ -valued process. For each $t \geq 0$ the random variable $S \star W(t)$ is generalized, stationary random field on \mathbb{R}^d with the spectral measure μ_t :

$$\mu_t(d\lambda) = \left[\int_0^t (s(\sigma, \beta(\lambda)))^2 d\sigma \right] \mu(d\lambda).$$

Theorem 5 ([11], Theorem 3) Assume that Hypothesis 4 holds. Then the process $S \star W(t)$ is function-valued for all $t \geq 0$ if and only if

$$\int_{\mathbb{R}^d} \left(\int_0^t (s(\sigma, \beta(\lambda)))^2 d\sigma \right) \mu(d\lambda) < +\infty, \quad t \geq 0.$$

If for some $\epsilon > 0$ and all $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} (\ln(1 + |\lambda|))^{1+\epsilon} (s(\sigma, \beta(\lambda)))^2 d\sigma \mu(d\lambda) < +\infty,$$

then, for each $t \geq 0$, $S \star W(t)$ is a sample continuous random field.

As we have already written, the Volterra equation (1) creates a big class of equations. In particular cases, when the operator A and the function a in the stochastic Volterra equation (2) are fixed, there is possible to obtain some additional regularity results. This is obvious because in particular cases we may use some extra features of solutions to (2). For instance, apparently well-known is integrodifferential equation which interpolates heat and wave equations, that is the Volterra equation (1), where $A = \Delta$, the Laplace operator and $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $0 \leq \alpha \leq 2$, where Γ is the gamma function. Recently, deterministic version was studied in detail by Fujita [6] and, independently, by Schneider and Wyss [14] and stochastic version of that integrodifferential equation was treated in [7] and [9]. In this paper we shall demonstrate numerical results obtained for that equation in the deterministic version and the stochastic one, as well.

In the theory of stochastic Volterra equations we are interested not only in the existence, uniqueness and regularity of solutions but in some asymptotics, too. The paper [8] is concerned with a limit measure of stochastic Volterra equation driven by very general noise in a form of a spatially homogeneous Wiener process, with values in the space of tempered distributions $S'(\mathbb{R}^d)$. That paper provides necessary and sufficient conditions for the existence of the limit measure and additionally, it gives a form of any limit measure.

Let us summarize the results cited above. The resolvent operators $S(t)$, $t \geq 0$, corresponding to the Volterra equation (1) do not form any semigroup. Therefore it is not possible to obtain such strong results as in the case of evolution equations with semigroup generators. In the case of Volterra equations one can not use *the fractional method of infinite dimensional stochastic calculus*. That method, used for demonstration of continuity with respect to t for convolutions with semigroups, enables to obtain only some estimates for the convolutions (4). Moreover, it is possible only in some special cases, see e.g. [10]. It is clear that the Volterra equation (1), in particular its stochastic version (2), is difficult to study. It results from the fact that equation (1) contains a wide class of equations. An essential role is played by the kernel function which, in general, is assumed to be a locally integrable function. As the function a and operator A determine the resolvent $S(t)$, $t \geq 0$, the type of the function a is very important. That is reason why it is so difficult to obtain in a general case the continuity of the convolution (4) and some other theoretical results. For more general convolution, significant in many applications, like $W^\psi(t) := \int_0^t S(t-\tau)\psi(\tau)dW(\tau)$, where ψ is an appropriate process, it becomes even far more difficult.

Therefore, in many cases a numerical support of theoretical (analytical) considerations is demanded. We need computations for obtaining estimates in

regularity results, choosing some parameters, choosing function a with required properties and for visualization of solutions obtained. Numerical analysis is particularly important when we are not able to obtain analytical results. Additionally, numerical schemes are especially useful for studying the asymptotics.

3 Galerkin method for deterministic Volterra equation

In this section we construct a scheme for numerical solution of the Volterra equation (2) without random part, that is, for

$$X(t, x) = \int_0^t a(t - \tau) A X(\tau, x) d\tau + X_0(x). \quad (9)$$

We shall consider the case when A is the Laplace operator. Denoting by $K(x, t, s) = a(t - s) \frac{d^2}{dx^2}$ we can write (9) in the standard form

$$X(x, t) = X_0(x) + \int_0^t K(x, t, s) X(x, s) ds. \quad (10)$$

In Galerkin method one introduces the complete set of orthonormal functions $\{\phi_j\}$, $j = 1, \dots, \infty$ on the interval $[0, t]$, that is fulfilling conditions

$$(\phi_i(t), \phi_j(t)) = \int_0^t \phi_i(\tau) \phi_j(\tau) d\tau = \delta_{ij}, \quad (11)$$

where (\cdot, \cdot) is the scalar product. The set $\{\phi_j\}$ spans a Hilbert space. The approximate solution is then postulated in the form of an expansion of the unknown true solution in the subspace H_n determined by n first basis functions

$$X_n(x, t) = \sum_{j=1}^n c_j(x) \phi_j(t). \quad (12)$$

Inserting (12) into (10) we obtain

$$X_n(x, t) = X_0(x) + \int_0^t K(x, t, s) X_n(x, s) ds + \varepsilon_n(x, t), \quad (13)$$

where the function $\varepsilon_n(x, t)$ represents the approximation error. From (13) we have

$$\varepsilon_n(x, t) = f_n(x, t) - X_0(x) - \int_0^t K(x, t, s) f_n(x, s) ds . \quad (14)$$

From (14) and (12) it can be written as

$$\varepsilon_n(x, \tau) = \sum_{k=1}^n c_k(x) \phi_k(\tau) - X_0(x) - \int_0^\tau K(x, \tau, s) \sum_{k=1}^n c_k(x) \phi_k(s) ds . \quad (15)$$

The coefficient functions $c_j(x)$ are determined by the requirement that the error function $\varepsilon_n(x, t)$ has to be orthogonal to the subspace H_n

$$(\phi_j(t), \varepsilon_n(x, t)) = 0 \quad j = 1, 2, \dots, n . \quad (16)$$

Then for $j = 1, 2, \dots, n$ the following equations hold

$$\begin{aligned} \int_0^t X_0(x) \phi_j(\tau) d\tau &= \int_0^t \left[\sum_{k=1}^n c_k(x) \phi_k(\tau) \right] \phi_j(\tau) d\tau \\ &\quad - \int_0^t \left[\int_0^\tau K(x, \tau, s) \sum_{k=1}^n c_k(x) \phi_k(s) ds \right] \phi_j(\tau) d\tau . \end{aligned} \quad (17)$$

The first integral on the r.h.s, due to (12), is very simple

$$\int_0^t \left[\sum_{k=1}^n c_k(x) \phi_k(\tau) \right] \phi_j(\tau) d\tau = \sum_{k=1}^n c_k(x) \int_0^t \phi_k(\tau) \phi_j(\tau) d\tau = c_j(x) . \quad (18)$$

The second one we calculate in our particular case, $K(x, \tau, s) = a(\tau - s) \frac{d^2}{dx^2}$ as follows

$$\begin{aligned} \int_0^t \left[\int_0^\tau \sum_{k=1}^n \frac{d^2 c_k(x)}{dx^2} a(\tau - s) \phi_k(s) ds \right] \phi_j(\tau) d\tau \\ = \sum_{k=1}^n \frac{d^2 c_k(x)}{dx^2} \int_0^t \left[\int_0^\tau a(\tau - s) \phi_k(s) ds \right] \phi_j(\tau) d\tau . \end{aligned} \quad (19)$$

Denoting by

$$g_j(x) = \int_0^t X_0(x) \phi_j(\tau) d\tau = X_0(x) \int_0^t \phi_j(\tau) d\tau, \quad \text{and} \quad (20)$$

$$a_{jk} = \int_0^t \phi_j(\tau) \left[\int_0^\tau a(\tau - s) \phi_k(s) ds \right] d\tau \quad (21)$$

we arrive at the set of coupled differential equations for the functions $c_j(x)$

$$g_j(x) = c_j(x) - \sum_{k=1}^n a_{jk} \frac{d^2 c_k(x)}{dx^2}. \quad (22)$$

This set can be solved numerically (aproximately) by discretization, on a grid $x_i = \{x_1, x_2, \dots, x_m\}$. Applying the difference form for the second derivative

$$\frac{d^2 c_k(x_i)}{dx^2} \approx \frac{1}{h^2} [c_k(x_{i-1}) - 2c_k(x_i) + c_k(x_{i+1})], \quad h = x_i - x_{i-1}$$

one obtains from (22) the following set of linear equations

$$g_j(x_i) = c_j(x_i) + \frac{1}{h^2} \sum_{k=1}^n a_{jk} [-c_k(x_{i-1}) + 2c_k(x_i) - c_k(x_{i+1})], \quad (23)$$

where $j = 1, 2, \dots, n$, $i = 1, 2, \dots, m$. Those equations can be written in the matrix form

$$\underline{A} \underline{c} = \underline{g}. \quad (24)$$

Here, $(N = n \times m)$ -dimensional vectors \underline{c} i \underline{g} have the following structures

$$\underline{c} = \begin{pmatrix} c_1(x_1) \\ \vdots \\ c_1(x_m) \\ c_2(x_1) \\ \vdots \\ c_2(x_m) \\ \dots \\ c_n(x_1) \\ \vdots \\ c_n(x_m) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}, \quad \underline{g} = \begin{pmatrix} g_1(x_1) \\ \vdots \\ g_1(x_m) \\ g_2(x_1) \\ \vdots \\ g_2(x_m) \\ \dots \\ g_n(x_1) \\ \vdots \\ g_n(x_m) \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix}, \quad (25)$$

where by C_i , G_i we denoted the consecutive m -dimensional blocks of vectors \underline{c} and \underline{g} , respectively. Then we can write the matrix \underline{A} in the block form

$$\underline{A} = \begin{pmatrix} [A_{11}] & \dots & [A_{1n}] \\ \vdots & \dots & \vdots \\ [A_{n1}] & \dots & [A_{nn}] \end{pmatrix}, \quad (26)$$

where every block is a tridiagonal matrix. The diagonal blocks have the following structure

$$[A_{ii}] = \begin{pmatrix} 1 + \frac{2}{h^2}a_{ii} & -\frac{1}{h^2}a_{ii} & 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{h^2}a_{ii} & 1 + \frac{2}{h^2}a_{ii} & -\frac{1}{h^2}a_{ii} & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{h^2}a_{ii} & 1 + \frac{2}{h^2}a_{ii} & -\frac{1}{h^2}a_{ii} & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{h^2}a_{ii} & 1 + \frac{2}{h^2}a_{ii} & -\frac{1}{h^2}a_{ii} \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{h^2}a_{ii} & 1 + \frac{2}{h^2}a_{ii} \end{pmatrix}, \quad (27)$$

and nondiagonal ones

$$[A_{ij}] = \begin{pmatrix} \frac{2}{h^2}a_{ij} & -\frac{1}{h^2}a_{ij} & 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{h^2}a_{ij} & \frac{2}{h^2}a_{ij} & -\frac{1}{h^2}a_{ij} & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{h^2}a_{ij} & \frac{2}{h^2}a_{ij} & -\frac{1}{h^2}a_{ij} & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{h^2}a_{ij} & \frac{2}{h^2}a_{ij} & -\frac{1}{h^2}a_{ij} \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{h^2}a_{ij} & \frac{2}{h^2}a_{ij} \end{pmatrix}. \quad (28)$$

The set of linear equations (24) can be solved by standard methods, for instance the LU decomposition [12].

4 Stochastic integral

The essential part of the mild solution (3) is the stochastic convolution (4). We present here a particular case when the resolvent $S(t)$ is known in analytical form. Let us focus the attention on stochastic Volterra equation (2) with the

function a in the form $a(t) = t^{\alpha-1}/\Gamma(\alpha)$. This is the integrodifferential equation [6], [8], [13]. For three particular cases, $\alpha = 0, 1, 2$ the analytical form of the resolvent S is known:

$$(S(t)v)(x) := \int_{-\infty}^{\infty} \phi_{\alpha}(t, x-y)v(y)dy = \int_{-\infty}^{\infty} \phi_{\alpha}(t, y)v(x-y)dy, \quad (29)$$

where the last form comes from the convolution property. We shall illustrate the applicability of our numerical algorithms with two cases of the above a functions, the case with $\alpha = 1$ and with $\alpha = 2$. Then the function ϕ_{α} in (29) takes the following form

$$\phi_1(t, x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \quad \text{for } \alpha = 1 \quad (30)$$

$$\phi_2(t, x) = \frac{1}{2}(\delta(t-x) + \delta(t+x)) \quad \text{for } \alpha = 2. \quad (31)$$

We assume the process in the form $W(t, x) = W_1(t)W_2(x)$. Then, the algorithm for an approximate construction of the stochastic integral (4) can be built in the following way. Let us introduce a time grid $\{t_i = i\tau : i = 0, 1, \dots, I\}$ on $[0, T]$, i.e. $\tau = T/I$ and next a finite sequence of independent random variables $\{\zeta_i\}$, $i = 1, 2, \dots, I$, with standard normal distribution. The approximation for the convolution (4) can be written in the form

$$\begin{aligned} \int_0^t S(t-s) dW(s, x) &= \sum_{i=0}^{I-1} S(t-s_i)[W(s_{i+1}, x) - W(s_i, x)] \\ &= \tau^{\frac{1}{2}} \sum_{i=0}^{I-1} \zeta_i \int_{-\infty}^{\infty} \phi_{\alpha}(t-s_i, x-y) W_2(y) dy \end{aligned} \quad (32)$$

For further specification we choose $W_2(x) = C X_0(x) = C e^{-x^2/4}$ (the constant C represents a 'strength' of stochastic forces). With this assumption, after performing the integral (32) for particular ϕ_{α} one obtains

$$\begin{aligned} \int_0^t S(t-s) dW(s, x) &= \\ C\tau^{\frac{1}{2}} \sum_{i=0}^{I-1} \zeta_i \frac{1}{\sqrt{1+t+s_i}} \exp\left(\frac{-x^2}{4(1+t+s_i)}\right) &\quad \text{for } \alpha = 1, \end{aligned} \quad (33)$$

and

$$\int_0^t S(t-s) dW(s, x) = \tag{34}$$

$$C\tau^{\frac{1}{2}} \sum_{i=0}^{I-1} \zeta_i^{\frac{1}{2}} \left[\exp\left(\frac{-(x-t+s_i)^2}{4}\right) + \exp\left(\frac{-(x+t-s_i)^2}{4}\right) \right] \quad \text{for } \alpha = 2.$$

These explicite forms were inserted into the numerical code. The sequence of independent random variables $\{\zeta_i\}$, $i = 1, 2, \dots, I$, with standard normal distribution was generated using subroutines **gasdev** and **ran1** from [12].

5 Numerical results

We illustrate the efficiency of the numerical approach on two examples of the function a , mentioned earlier. As the initial value of the $X(t=0, x) = X_0(x)$ we chose the Gaussian $X_0(x) = e^{-x^2/4}$. The grid in x variable contained $m = 150$ intervals with $h = 0.2$, covering the interval $x \in [-15, 15]$. The dimension of the approximation subspace in the Galerkin method was chosen as $n = 8$. The resulting dimension of the matrix \mathcal{A} was then 1208×1208 and calculations were performed up to $T = 6$.

In fig. 1 we show the errors of the numerical solutions to the deterministic equation (9) obtained in cases $\alpha = 1$ (top), and $\alpha = 2$ (bottom). In both cases the errors are relatively small.

The top part of the fig. 2 displays the solution of the deterministic Volterra equation 1 for the case $\alpha = 1$ as function of time, $t \in [0, 6]$. The bottom part of the fig. 2 shows the example of a single stochastic trajectory (i.e. the sum of deterministic solution and stochastic integral) for the same case.

The fig. 3 present the corresponding solutions for the case $\alpha = 2$. For stochastic convolutions the value of the constant C was chosen to be $C = 0.1$.

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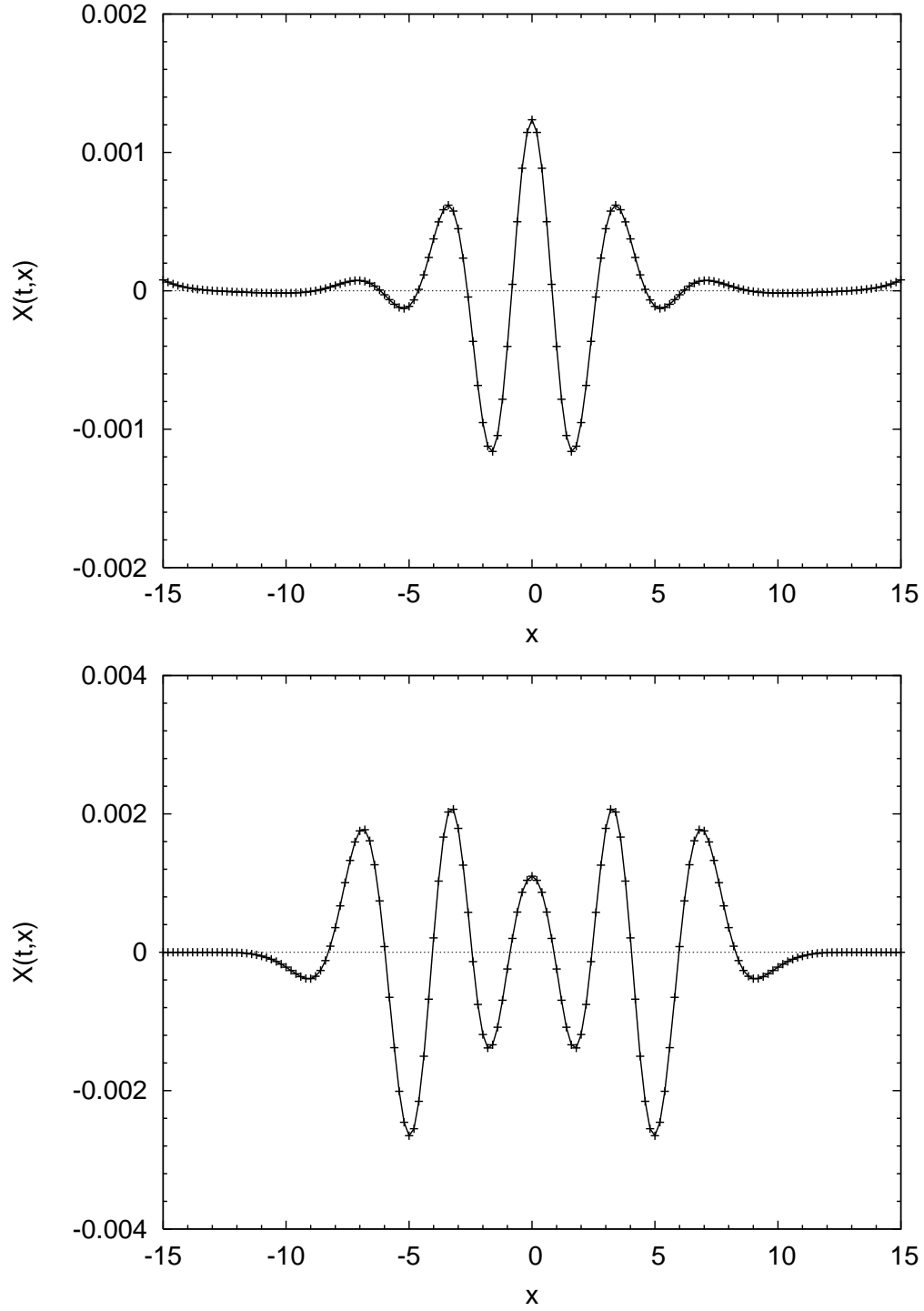


Fig. 1. Errors of the numerical solutions to deterministic Volterra equation (9) at $t = 6$. Top: $\alpha = 1$ case, bottom: $\alpha = 2$ case. The symbols represent the difference between the exact (analytical) solution and the numerical solution at given grid points.

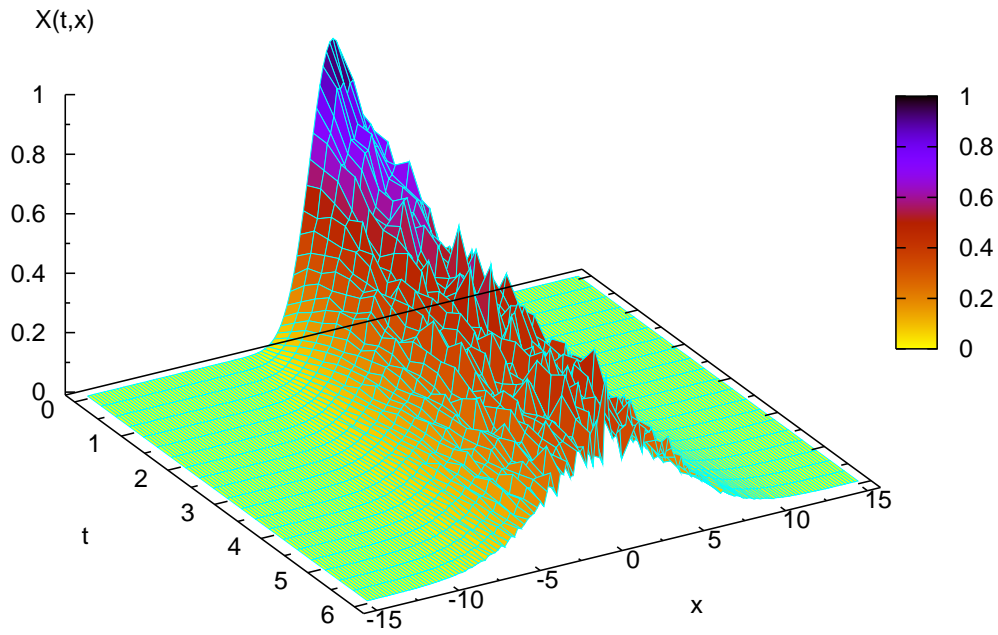
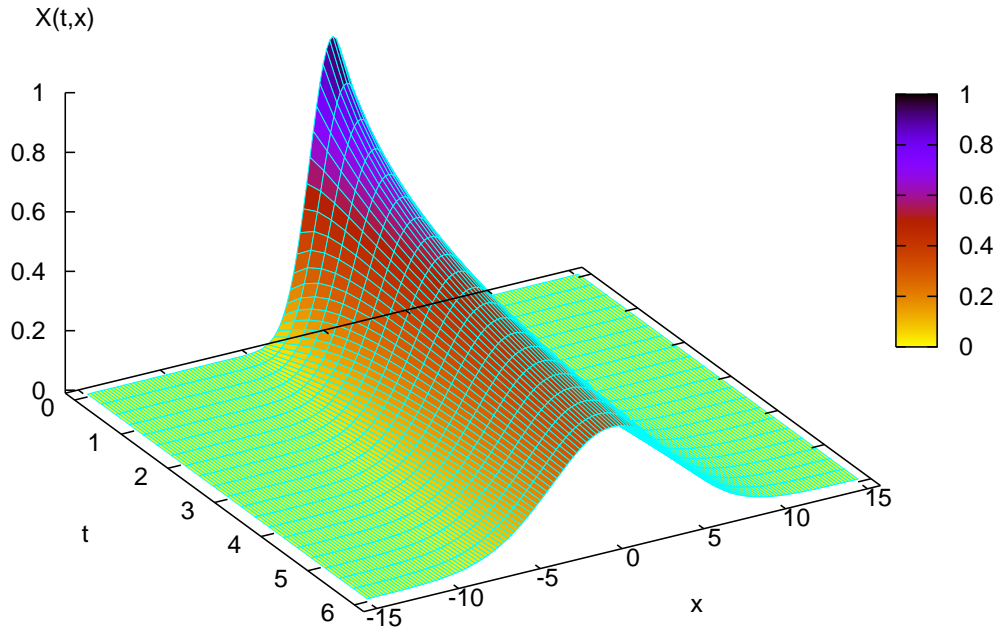


Fig. 2. Numerical solution to Volterra equation with $\alpha = 1$ for $t \in [0, 6]$: the deterministic solution (top) and the stochastic one (bottom).

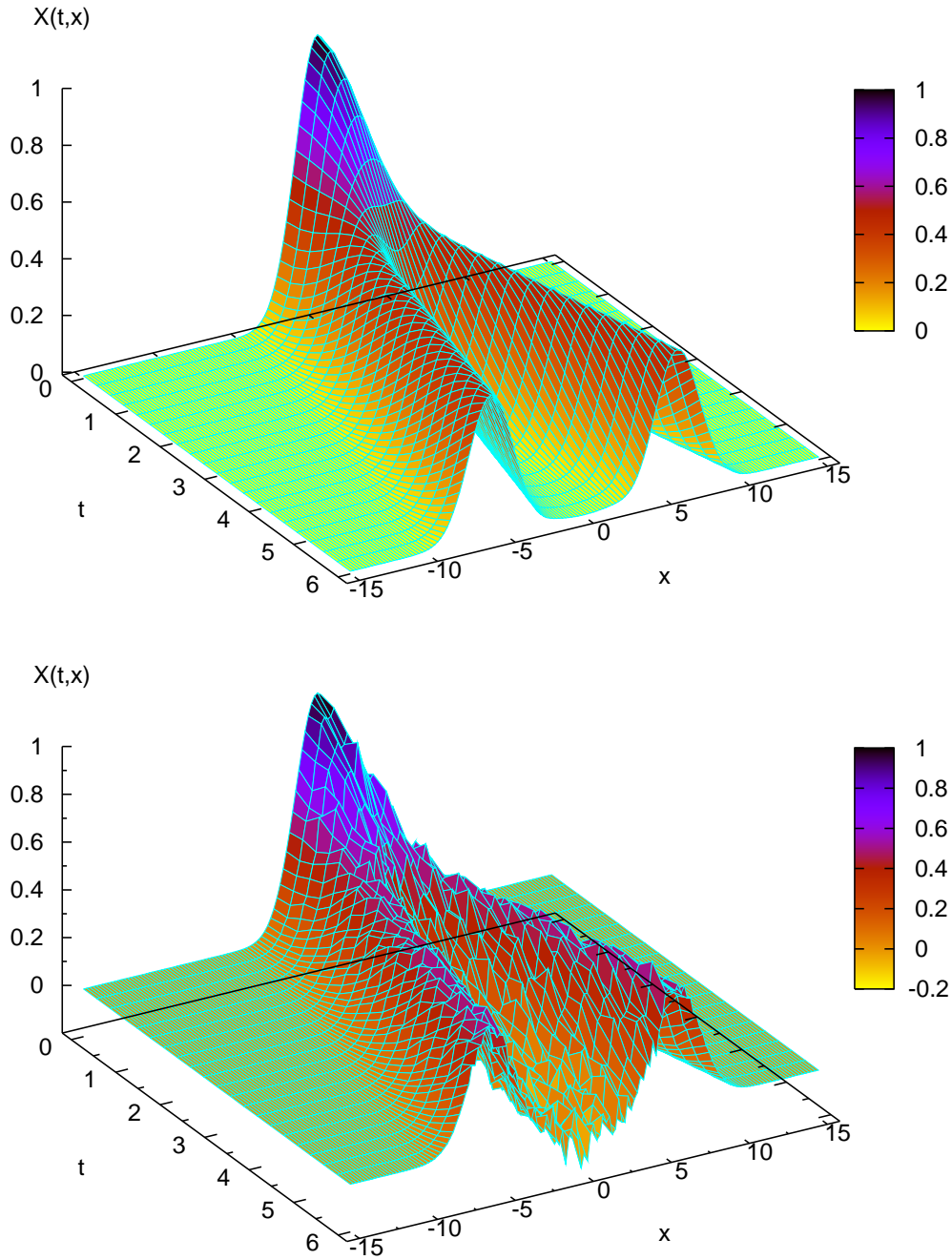


Fig. 3. Numerical solution to Volterra equation with $\alpha = 2$ for $t \in [0, 6]$: the deterministic solution (top) and the stochastic one (bottom).